



中山大學

SUN YAT-SEN UNIVERSITY

# Circuit Quantum Electrodynamics

Shangrun Lu <sup>1,2</sup>

Ziyu He <sup>1,2</sup>

<sup>1</sup> School of Physics

<sup>2</sup> Sun Yat-sen University, GuangZhou, China

2023.10.16

# CONTENTS

01

Introduction

02

LC Resonator

03

Fritz London Theory

04

Josephson Junction

05

Superconducting Circuits



中山大學

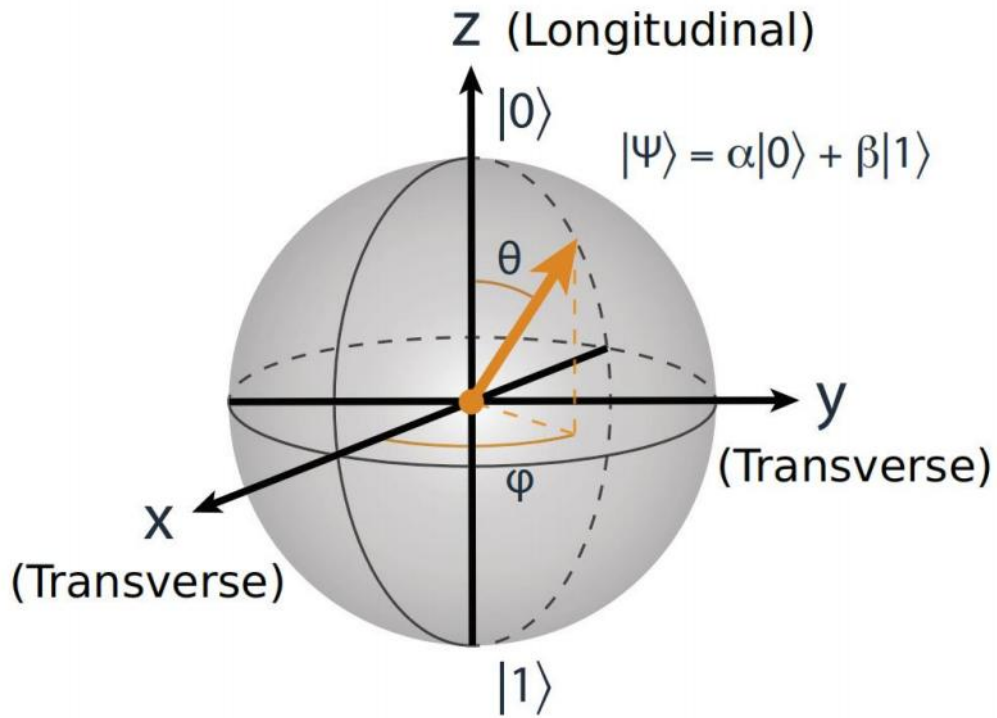
SUN YAT-SEN UNIVERSITY

Part.01

# Introduction



• Bloch Sphere



Bloch sphere

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

	Circuit	Properties	Dominant noise
Charge qubit		$E_J/E_C < 1$ Controlled by $V_g$ .	Charge fluctuations; mainly 1/f noise.
		$E_J/E_C < 1$ Controlled by both $V_g$ and $\Phi_e$ .	
Flux qubit		$E_J/E_C > 1$ Controlled by $\Phi_e$ .	Flux fluctuations; mainly 1/f noise.
		$E_J/E_C > 1$ $0.5 < \alpha < 1$ Controlled by $\Phi_e$ .	
Phase qubit		$E_J/E_C \gg 1$ Controlled by $I_e$ .	Flux fluctuations; mainly 1/f noise.

Different types of superconducting qubits



中山大學

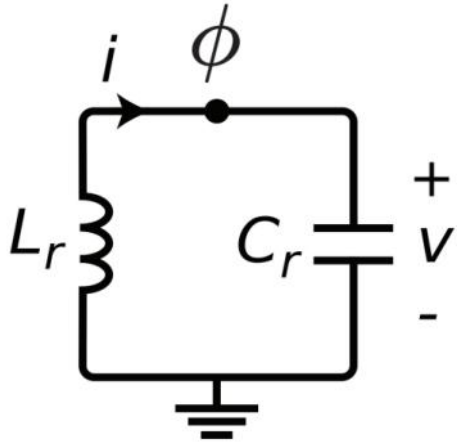
SUN YAT-SEN UNIVERSITY

Part.02

# LC Resonator



•Quantum Harmonic Oscillator



LC circuit

$$V = \frac{d\Phi}{dt}$$

$$E_L = \frac{\Phi^2}{2L}$$

$$E_C = \frac{C\dot{\Phi}^2}{2}$$

$$L = \frac{C\dot{\Phi}^2}{2} - \frac{\Phi^2}{2L}$$

$$p_\Phi = \frac{\partial L}{\partial \dot{\Phi}} = C\dot{\Phi} = Q$$

$$\{\Phi, Q\} = 1$$



$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

•Quantization

Harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$[\hat{x}, \hat{p}] = i\hbar$$



LC Circuit

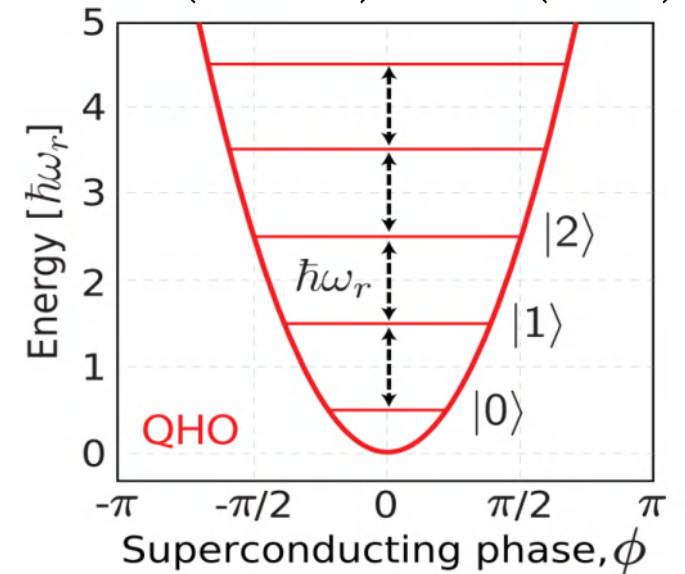
$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar Z_r}} (\hat{\Phi} - iZ_r\hat{Q})$$

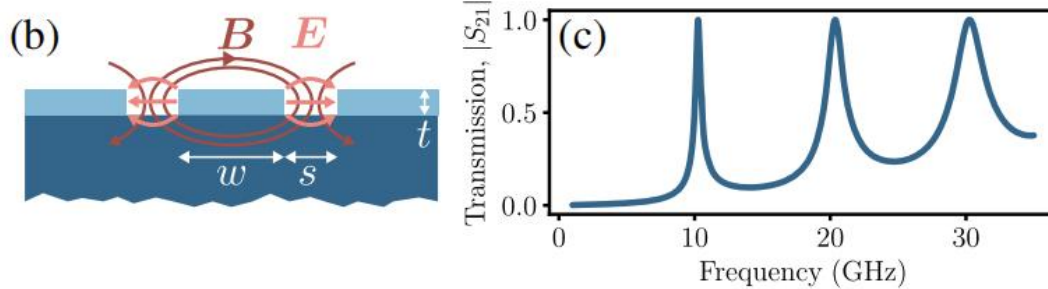
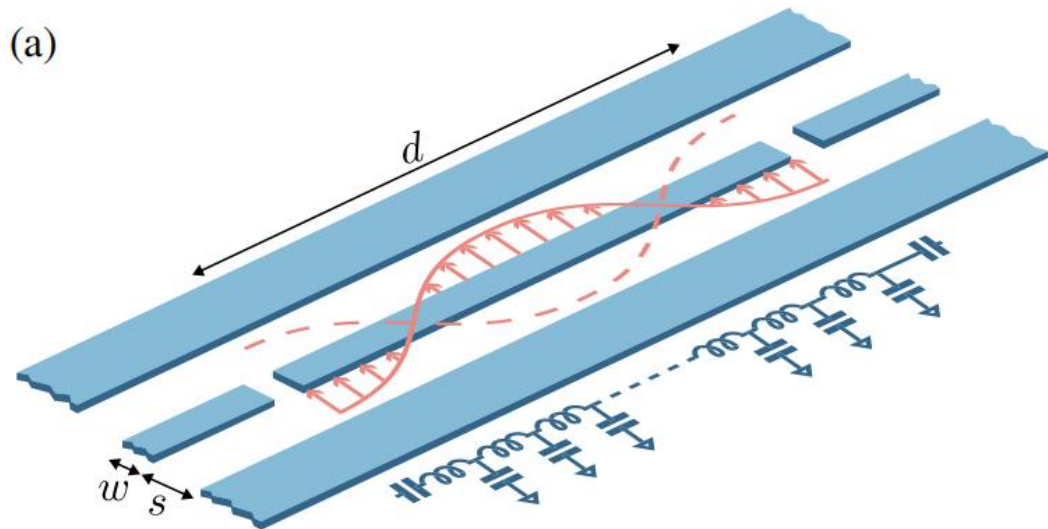
$$a = \frac{1}{\sqrt{2\hbar Z_r}} (\hat{\Phi} + iZ_r\hat{Q})$$

$$\hat{H} = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) = \hbar\omega \left( \hat{n} + \frac{1}{2} \right)$$

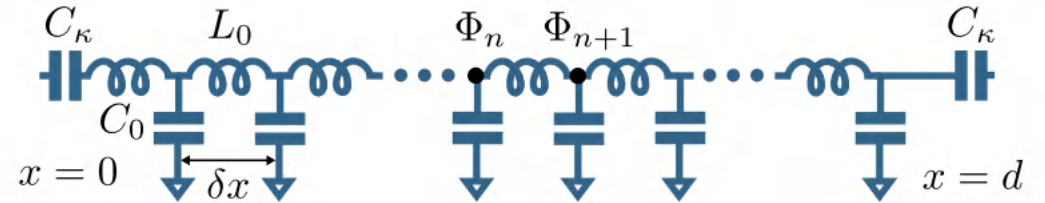


Energy levels

•2D Resonator



Coplanar waveguide resonator



Telegrapher model

$$C_0 = c_0 \delta x$$

$$L_0 = l_0 \delta x$$

$$H = \sum_{n=0}^{N-1} \left[ \frac{Q_n^2}{2C_0} + \frac{(\Phi_{n+1} - \Phi_n)^2}{2L_0} \right]$$

↓  $\delta x \rightarrow 0$

$$H = \int_0^d dx \left\{ \frac{1}{2c_0} Q(x)^2 + \frac{1}{2l_0} [\partial_x \Phi(x)]^2 \right\}$$

Hamiltonian equation:

$$v_0^2 \frac{\partial^2 \Phi(x, t)}{\partial x^2} - \frac{\partial^2 \Phi(x, t)}{\partial t^2} = 0$$

$$\Phi(x, t) = \sum_{m=0}^{\infty} u_m(x) \Phi_m(t)$$

$$H = \sum_{m=0}^{\infty} \left[ \frac{Q_m^2}{2Cr} + \frac{1}{2} C_r \omega_m^2 \Phi_m^2 \right]$$

$$= \sum_{m=0}^{\infty} \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m$$



中山大學

SUN YAT-SEN UNIVERSITY

Part.03

# Fritz London Theory





## •Fritz London Theory

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m^*} (-i\hbar \nabla + q^* \vec{A})^2 \psi + q^* \phi \psi + V \psi$$

$$\vec{J} = \frac{iq^*\hbar}{2m^*} (\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{q^{*2}}{m^*} \psi^* \psi \vec{A}$$

$$\psi = \sqrt{n(\vec{r}, t)} e^{i\theta(\vec{r}, t)}$$

$$\vec{J} = \frac{-2ne^2}{m_e} \left( \frac{\hbar}{2e} \nabla \theta + \vec{A} \right)$$

## •London First Equation

$$\frac{\partial \vec{J}}{\partial t} = \frac{-2ne^2}{m_e} \left( \frac{\hbar}{2e} \nabla \frac{\partial \theta}{\partial t} + \frac{\partial \vec{A}}{\partial t} \right)$$

$$\frac{\partial \theta}{\partial t} = \frac{-\epsilon}{\hbar} = -\frac{1}{\hbar} \left( \frac{m_e}{2ne^2} \frac{J^2}{2n} - 2e\phi \right)$$

$$\frac{\partial \vec{J}}{\partial t} = \frac{-2ne^2}{m_e} \left( \nabla \phi + \frac{\partial \vec{A}}{\partial t} \right) + ne \nabla \left( \frac{J^2}{4n^2 e^2} \right) \approx \frac{-2ne^2}{m_e} \vec{E}$$

## •London Second Equation

$$\nabla \times \vec{J} = \frac{-2ne^2}{m_e} \left( \frac{\hbar}{2e} \nabla \times \nabla \theta + \nabla \times \vec{A} \right) = \frac{-2ne^2}{m_e} \vec{B}$$



•Fritz London Theory

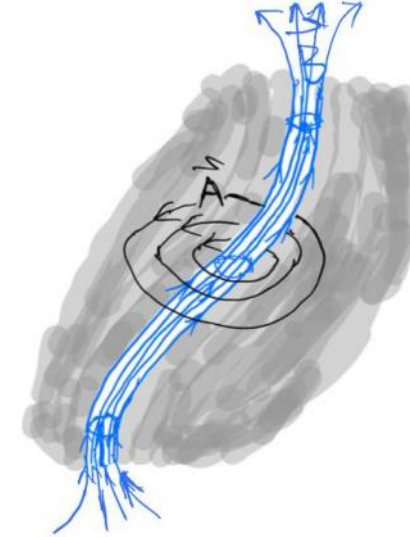
$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m^*} (-i\hbar \nabla + q^* \vec{A})^2 \psi + q^* \phi \psi + V \psi$$

$$\vec{J} = \frac{iq^*\hbar}{2m^*} (\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{q^{*2}}{m^*} \psi^* \psi \vec{A}$$

$$\psi = \sqrt{n(\vec{r}, t)} e^{i\theta(\vec{r}, t)}$$

$$\vec{J} = \frac{-2ne^2}{m_e} \left( \frac{\hbar}{2e} \nabla \theta + \vec{A} \right)$$

•Flux Quantum



$$0 = \oint \vec{J} \cdot d\vec{l} = \frac{-2ne^2}{m_e} \left( \oint \frac{\hbar}{2e} \nabla \theta \cdot d\vec{l} + \oint \vec{A} \cdot d\vec{l} \right)$$

$$\Phi = \oint \vec{A} \cdot d\vec{l} = - \oint \frac{\hbar}{2e} \nabla \theta \cdot d\vec{l} = \frac{\hbar}{2e} 2k\pi$$



中山大學

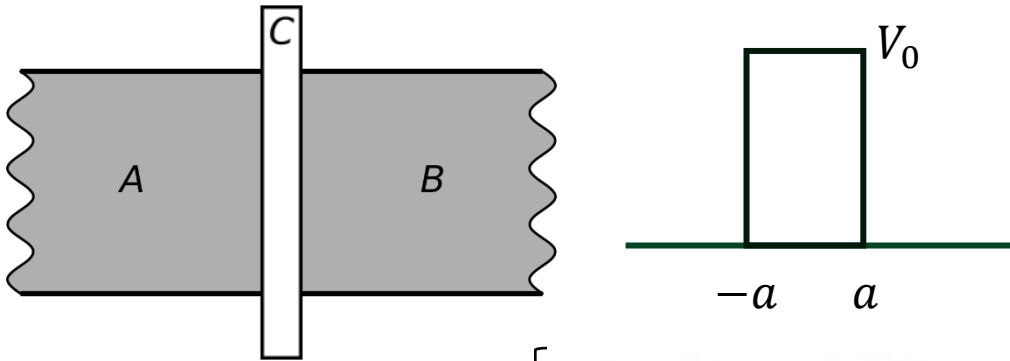
SUN YAT-SEN UNIVERSITY

Part.04

# Josephson Junction



• Josephson Junction



$$J_0 = \frac{-n\hbar e}{m_e} \nabla \theta(\pm a, t) \quad \left\{ \begin{array}{l} \psi(-a, t) = \sqrt{n_1} e^{i\theta_1} \\ \psi(+a, t) = \sqrt{n_2} e^{i\theta_2} \end{array} \right.$$

$$\epsilon_0 \psi = -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V \psi$$

$$\psi(x) = C_1 \cosh(x/b) + C_2 \sinh(x/b)$$

$$b = \sqrt{\frac{\hbar^2}{2m^*(V_0 - \epsilon_0)}} \quad \left\{ \begin{array}{l} C_1 = \frac{\sqrt{n_2^*} e^{i\theta_2} + \sqrt{n_1^*} e^{i\theta_1}}{2 \cosh(a/b)} \\ C_2 = \frac{\sqrt{n_2^*} e^{i\theta_2} - \sqrt{n_1^*} e^{i\theta_1}}{2 \sinh(a/b)} \end{array} \right.$$

$$J = \frac{e\hbar \sqrt{n_1 n_2}}{m_e b \sinh(\frac{2a}{b})} \sin(\theta_1 - \theta_2)$$

$$\vec{A}' = \vec{A} + \xi \quad \phi' = \phi - \frac{\partial \xi}{\partial t} \quad \theta' = \theta - \frac{2e}{\hbar} \xi$$

$$\varphi \equiv \theta_1 - \theta_2 + f = \theta_1 - \theta_2 - \frac{2e}{\hbar} \int_{12} \vec{A} \cdot d\vec{l}$$

$$J = \frac{e\hbar \sqrt{n_1 n_2}}{m_e b \sinh(\frac{2a}{b})} \sin(\varphi)$$

$$\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} U$$

$$I = I_C \sin \varphi$$



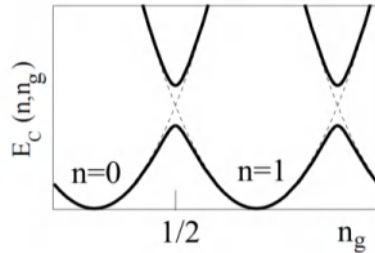
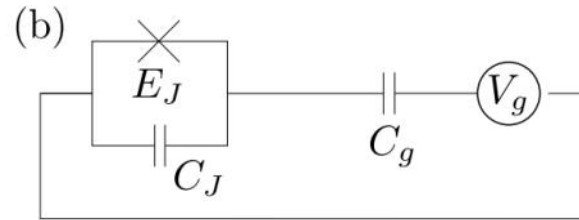
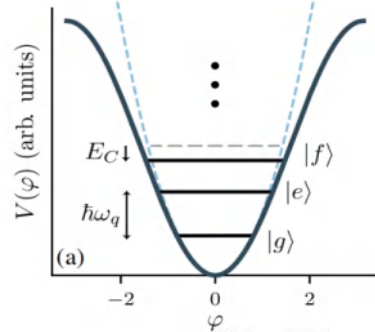
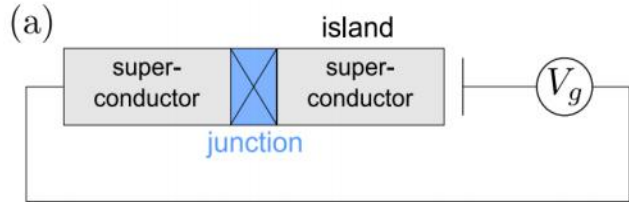
中山大學

SUN YAT-SEN UNIVERSITY

Part.05

# Superconducting Circuits

• Charge Qubit



Charge qubit

Energy levels

$$L = \frac{1}{2} C_J (\dot{\Phi})^2 + \frac{1}{2} C_g (\dot{\Phi} - V_g)^2 + E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$



$$p = 2en$$

$$n_g = C_g V_g / 2e \quad H = E_C (n - n_g)^2 - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$

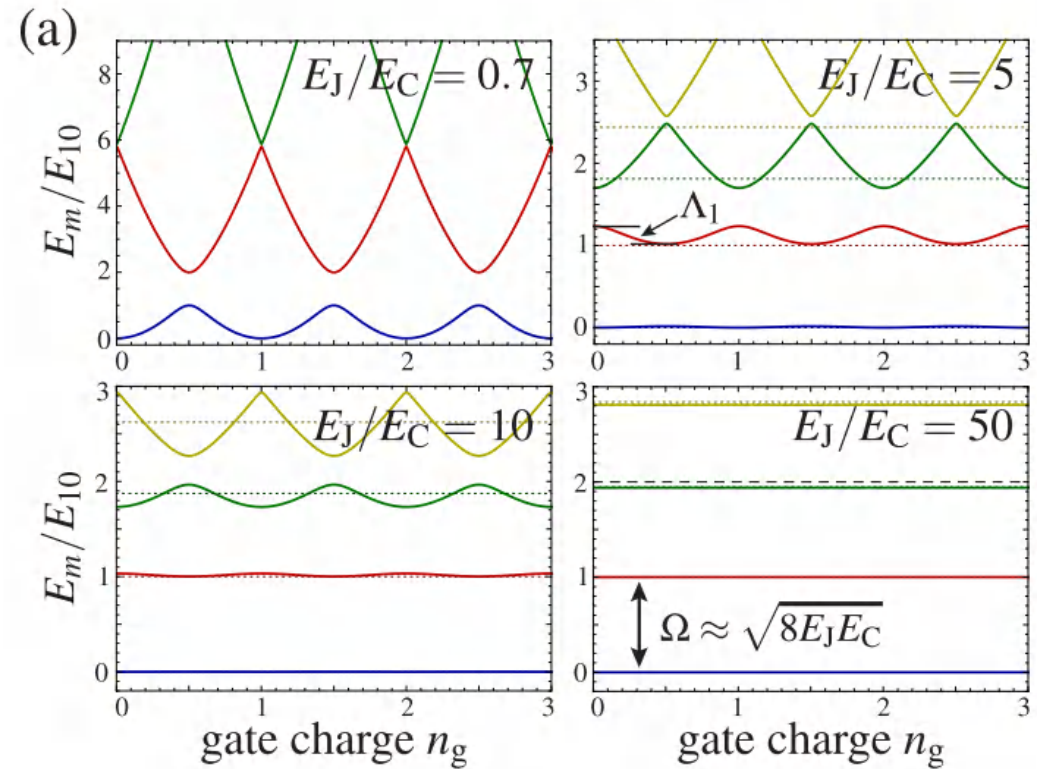
• Transmon Qubit

Loss of anharmonicity  $\sim (E_J/E_C)^{-1/2}$

$E_J/E_C \ll 1$ : charge qubit

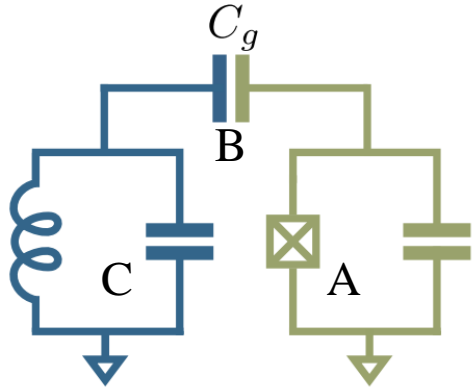
$E_J/E_C \sim 1$ : quantronium

$E_J/E_C \gg 1$ : transmon qubit



Energy spectrum at different energy ratios

• Exchange Interaction

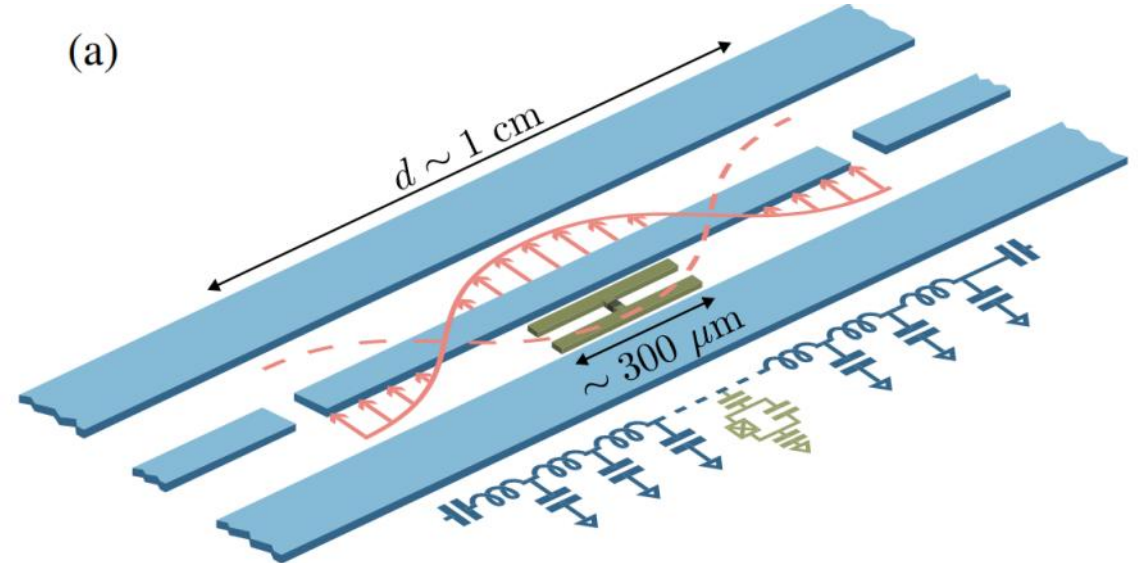


$$\left\{ \begin{array}{l} \dot{\Phi}_C + \dot{\Phi}_B - \dot{\Phi}_A = 0 \\ I_A = -I_B \\ I_B = I_C \\ I_C = C\ddot{\Phi}_C + \frac{\Phi_C}{L} \\ I_B = C_g\ddot{\Phi}_B \end{array} \right.$$

Transmon and Oscillator

$$\begin{aligned} H &\approx \frac{1}{2C_\Sigma C} [CQ_A^2 + C_\Sigma Q_C^2 + C_g(Q_A + Q_C)^2] + V \\ &= \frac{Q_A^2 + \frac{C_g}{C}(Q_A + Q_C)^2}{2C_\Sigma} + \frac{\Phi^2}{2L} + \frac{Q_C^2}{2C} - E_J \cos(\phi_A) \\ &\approx \frac{\left(Q_A + \frac{C_g}{C}Q_C\right)^2}{2C_\Sigma} + \frac{\Phi^2}{2L} + \frac{Q_C^2}{2C} - E_J \cos(\phi_A) \end{aligned}$$

(a)



Transmon Qubit Coupled to 2D Transmission-line Resonator

$$8E_r \hat{n} \cdot \hat{n}_r = -\hbar\omega_r \frac{C_g}{C_\Sigma} \sqrt{\frac{\pi Z_r}{R_k} \left(\frac{E_J}{2E_C}\right)^{\frac{1}{4}}} \text{Coupling factor: } g (a^\dagger - a)(b^\dagger - b)$$

$$\hat{H} = \hbar\omega_r a^\dagger a + \hbar\omega_q b^\dagger b - \frac{E_C}{2} b^\dagger b^\dagger b b - \hbar g \underbrace{(ab + a^\dagger b^\dagger)}_{\text{CRT}} + \underbrace{ab^\dagger + a^\dagger b}_{\text{RWA}}$$



中山大學

SUN YAT-SEN UNIVERSITY

Part.00

Appendix





## •References

- [1] Krantz, Philip et al. “A quantum engineer's guide to superconducting qubits.” Applied Physics Reviews (2019): n. pag.
- [2] Xiang, Z.-L., Ashhab, S., You, J.Q., Nori, F.. Hybrid quantum circuits: Superconducting circuits interacting with other quantum systems[J]. Reviews of Modern Physics, 2013, 85(2).
- [3] Windin G, Shumeiko V S. Superconducting Quantum Circuits, Qubits and Computing[J]. Arxiv Preprint Cond, 2005.
- [4] Alexandre Blais, Arne L. Grimsmo, S. M. Girvin, and Andreas Wallraff. Circuit quantum electrodynamics[J]. Rev. Mod. Phys. 93, 025005.
- [5] Johansson R. Quantum mechanics in superconducting circuits and nanomechanical devices[J]. Chalmers University of Technology, 2009.
- [6] Viehmann O .Multi-qubit circuit quantum electrodynamics[J]. 2013.
- [7] Makhlin, Yuriy and Schoen, Gerd and Shnirman, Alexander. Quantum-state engineering with Josephson-junction devices[J]. Rev.Mod.Phys. 73 (2001) 357-400.
- [8] London, F., & London, H. (1935). The electromagnetic equations of the supraconductor. Proceedings of the Royal Society of London. Series A-Mathematical and Physical Sciences, 149(866), 71-88.
- [9] 固体理论[M/OL]. 高等教育出版社, 2002. <https://books.google.com.tw/books?id=bSvGAAAACAAJ>